MAT-499 UG Thesis Report: Chess Combinatorics

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Abstract

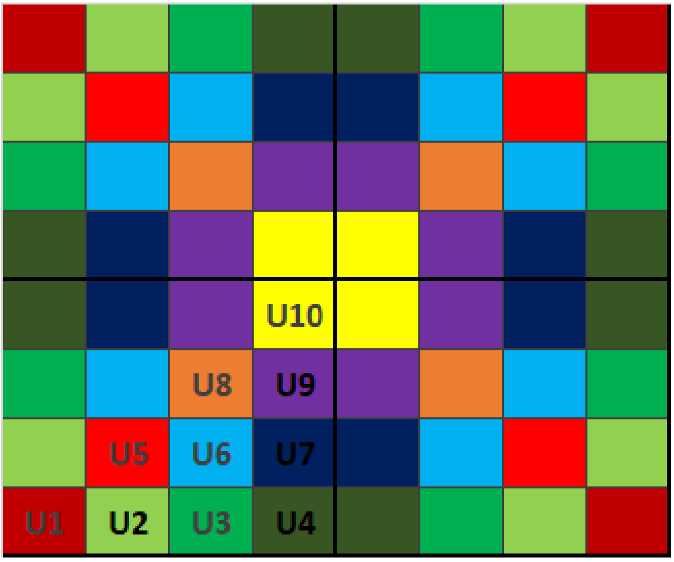
Through this report, we observe the moves, paths and walks of different chess pieces on an empty chess board, the traversals of each chess piece can be represented as a graph with vertices and edges. We will observe the results for pieces like: king, queen, knight and bishop in 4x4 board as a sample then the 8x8 standard board and attempt to find results for extended chess boards of varying dimensions specifically for the rook.

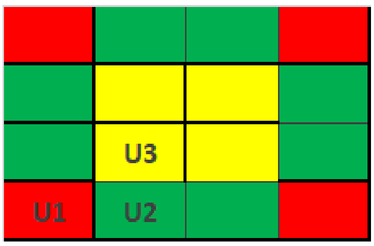
Introduction

Main objectives will be to find the number of ways: a king, queen, knight and bishop can traverse in an 4x4 & 8x8 chess board with special emphasis on the rook which is extended for any arbitrary length and dimension. Following concepts will be used to approach our problem:

- Binomial Theorem: For any two reals, $(x+y)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} y^r$
- Walks in Graph Theory: No. of closed walks of n-length in K_N (Result 1) $= \frac{(N-1)^n + (N-1)(-1)^n}{N}$, where K_N is a complete graph with N vertices.
- Chess board, its dimensions, geometry and the symmetries of its squares
- Matrices (or adjacency matrices) and solutions of simultaneous recursive equations, to find the n^{th} power of a matrix we need to diagonalize the matrix, where the diagonal entries will contain the eigen values of the matrix, for any diagonal matrix D: $A = VDV^{-1} \Rightarrow A^n = VD^nV^{-1}$

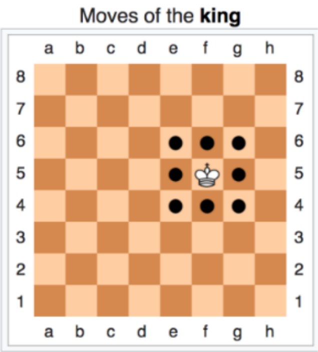
Chess Board

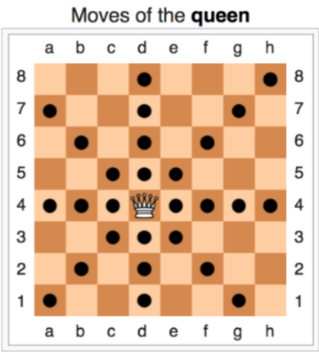


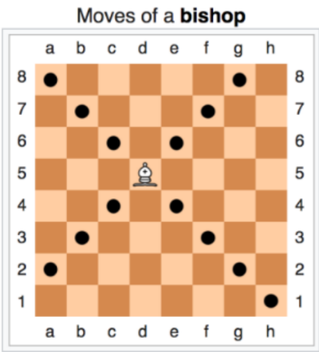


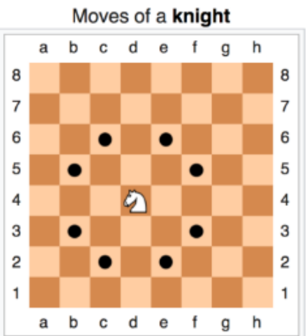
The chess board used will be: 8 x 8 board containing 64 squares with 10 distinct squares and a 4x4 board with 16 squares and 3 distinct squares, where the rest of the squares are similar due to symmetry, as shown above where 10 and 3 of them are coloured respectively. For a square of dimension: $L \times L$, it will be: $\frac{m(m+1)}{2}$,Where $m = \text{Floor}\left[\frac{L+1}{2}\right]$, m is the greater half of L (m is 2 when L is 3 or 4, m is 4 when L is 7 or 8 and m is 5 when L is 9 or 10). In a k-dimensional board, used for the rook related problem, the Dimension would be of the form: $L_1 \times L_2 \times ... \times L_k$, because the squares of the board does not affect the movement of a rook, every square is similar for a rook, hence symmetries and finding identical squares for a higher dimensional board is not required as of now.

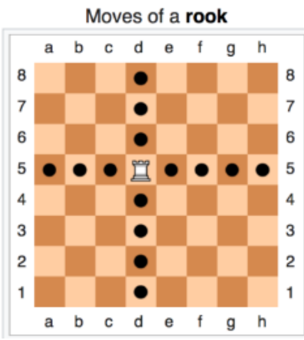
Pieces in the board











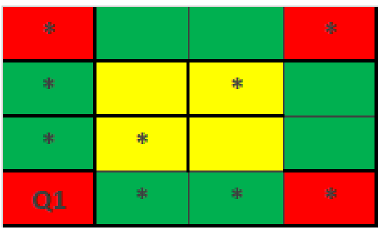
King: In the corner it can move in 3 ways, in edges 5 ways, otherwise 8,
Queen: in the first layer(U1-U4) it can move in 21 ways, 2nd layer(U5-U7) it can move in 23 ways, 3rd layer(U8,U9) it can move in 25 ways, 4th layer(U10) it can move in 27 ways,
Knight: in corners 2 only and centre up to 8, varies from 2,3,4,6 and 8,
Bishop: 7 in the 1st layer, 9 in 2nd , 11 in 3rd and 13 in the 4th layer respectively,
Rook can move 14 ways for all the squares.
For the 4 by 4 simply truncate the board to get its moves, the king will be least affected whereas the knight will be most affected.

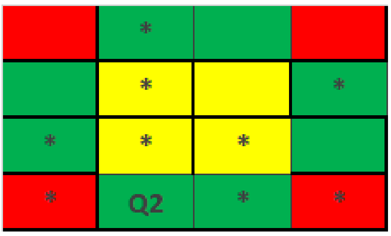
Traversing the board with Chess pieces

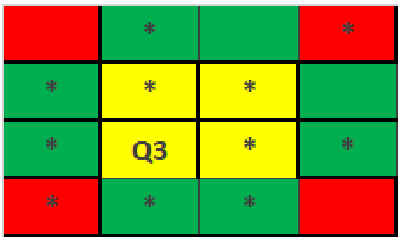
Queen, King, Knight and Bishop:

<u>*4x4 board:*</u> We first find ways a queen can traverse the 4x4 board in n moves, Let Q1(n) be no. of ways a queen can traverse the board in n moves starting from any fixed corner, similarly Q2(n) for edges and Q3(n) for centre squares

The queen's path in the corner edge and centre will be:







The queen in Q1 can move in 9 ways per move to 3 red(Q1), 4 green(Q2) and 2 yellow(Q3) respectively,
The queen in Q2 can move in 9 ways per move to 2 red(Q1), 4 green(Q2) and 3 yellow(Q3) respectively,
The queen in Q3 can move in 11 ways per move to 2 red(Q1), 6 green(Q2) and 3 yellow(Q3) respectively
The colouring groups by Q1,Q2,Q3 is all due to symmetry of the chess board, similar patterns and results are obtained if different corners, edges and centres are chosen.
One can obtain a set of recursive relations from the above:

$$Q_1(n)=(3Q_1+4Q_2+2Q_3)(n-1)$$

$$Q_2(n)=(2Q_1+4Q_2+3Q_3)(n-1)$$

$$Q_3(n)=(2Q_1+6Q_2+3Q_3)(n-1)$$

Using matrix relations we obtain:

$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}(n) = \begin{bmatrix} 3 & 4 & 2 \\ 2 & 4 & 3 \\ 2 & 6 & 3 \end{bmatrix}\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}(n-1) \Rightarrow \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}(n) = \begin{bmatrix} 3 & 4 & 2 \\ 2 & 4 & 3 \\ 2 & 6 & 3 \end{bmatrix}^n \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}(0) = \begin{bmatrix} 3 & 4 & 2 \\ 2 & 4 & 3 \\ 2 & 6 & 3 \end{bmatrix}^n \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

,where Q1,Q2,Q3 at n=0 is equal to 1

Let the above be represented as: $Q(n) = Q(1 \quad 1 \quad 1)^T$

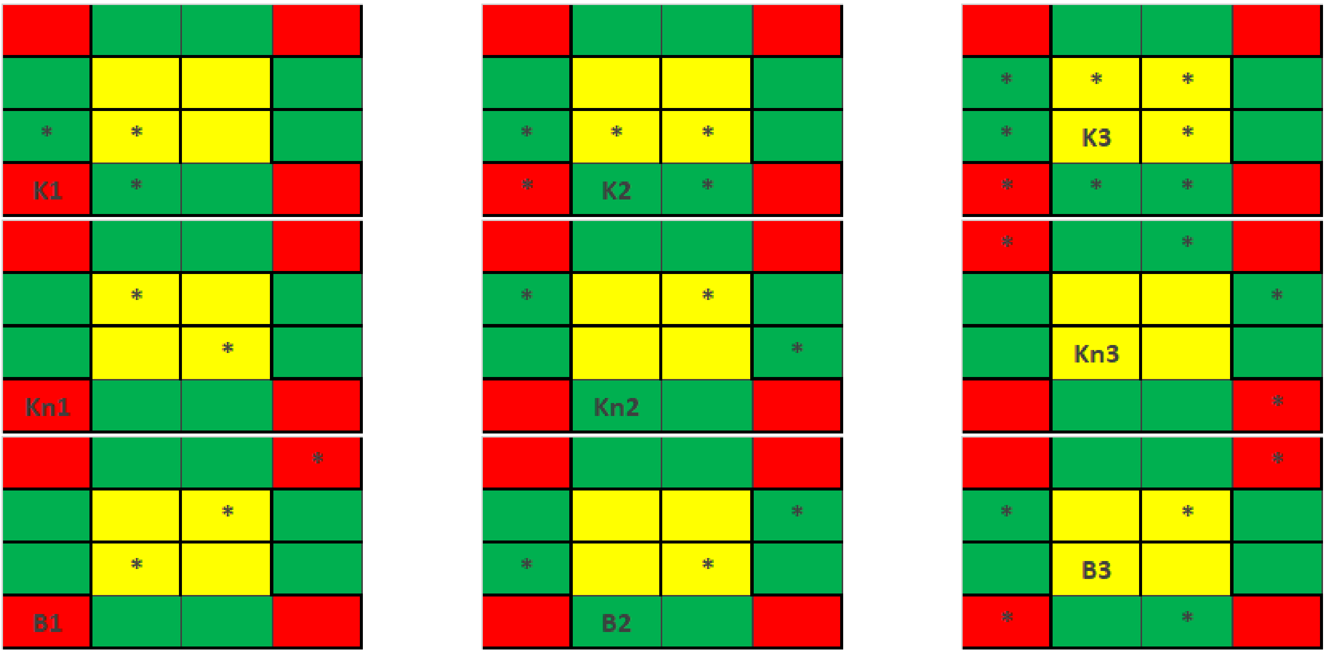
Using matrix calculators[2] we get,

the characteristic polynomial as: $|xI - Q| = x^3 - 10x^2 + 3x + 10$, with largest eigenvalue $\rho(Q) \approx 9.5778$ and the corresponding values of Qi will be:

$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}(n) \approx (9.5778)^n \begin{bmatrix} 0.9251 \\ 0.9482 \\ 1.1462 \end{bmatrix}$$

The above is obtained by computing: $Q(n) \approx \rho(Q)^n \left(\rho(Q)^{-1} Q\right)^{100} (1 \quad 1 \quad 1)^T$, as n approaches larger values the terms with powers less than the spectral radius become significantly smaller and negligible impact to the magnitude of Q. The actual value can be computed by taking powers of Q or by diagonalizing it.

Similarly for the rest of the pieces we get:



We get the following recursions and matrix relations:

$$\begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix}(n) = \begin{bmatrix} 0K_1 + 2K_2 + 1K_3 \\ 1K_1 + 2K_2 + 2K_3 \\ 1K_1 + 4K_2 + 3K_3 \end{bmatrix}(n-1) = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 4 & 3 \end{bmatrix}^n \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} Kn_1 \\ Kn_2 \\ Kn_3 \end{bmatrix}(n) = \begin{bmatrix} 0Kn_1 + 0Kn_2 + 2Kn_3 \\ 0Kn_1 + 2Kn_2 + 1Kn_3 \\ 2Kn_1 + 2Kn_2 + 0Kn_3 \end{bmatrix}(n-1) = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 2 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}(n) = \begin{bmatrix} 1B_1 + 0B_2 + 2B_3 \\ 0B_1 + 2B_2 + 1B_3 \\ 2B_1 + 2B_2 + 1B_3 \end{bmatrix}(n-1) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 2 & 1 \end{bmatrix}^n \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

With char-polynomials:

$$|xI - K| = x(x^2 - 5x - 5), |xI - Kn| = x^3 - 2x^2 - 6x + 8, |xI - B| = x^3 - 4x^2 - x + 8$$

And the approximate result to be:

$$\begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix}(n) \approx (5.8541)^n \begin{bmatrix} 0.5236 \\ 0.8472 \\ 1.3709 \end{bmatrix}, \begin{bmatrix} Kn_1 \\ Kn_2 \\ Kn_3 \end{bmatrix}(n) \approx (3.1028)^n \begin{bmatrix} 0.728 \\ 1.024 \\ 1.129 \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}(n) \approx (3.6813)^n \begin{bmatrix} 0.968 \\ 0.771 \\ 1.297 \end{bmatrix}$$

The above is obtained in similar ways was it for the queen using matrix calculators.

<u>8x8 board:</u> We find ways a queen can traverse the 8x8 board in n moves, or the number of walks of n-length, for walks with fixed starting and ending points we may simply use respective probabilities (covered soon).
If we define 10 functions Q1, ... ,Q10 representing no. of such ways to traverse the board starting from U1, ... ,U10 respectively, then we get:

$$\begin{aligned}
Q_1(n) &= (3Q_1+4Q_2+4Q_3+4Q_4+2Q_5+0Q_6+0Q_7+2Q_8+0Q_9+2Q_{10})(n\text{-}1) \\
Q_2(n) &= (2Q_1+4Q_2+2Q_3+2Q_4+2Q_5+4Q_6+2Q_7+0Q_8+2Q_9+1Q_{10})(n\text{-}1) \\
Q_3(n) &= (2Q_1+2Q_2+4Q_3+2Q_4+1Q_5+2Q_6+2Q_7+2Q_8+4Q_9+0Q_{10})(n\text{-}1) \\
Q_4(n) &= (2Q_1+2Q_2+2Q_3+4Q_4+0Q_5+2Q_6+4Q_7+1Q_8+2Q_9+2Q_{10})(n\text{-}1) \\
Q_5(n) &= (2Q_1+4Q_2+2Q_3+0Q_4+3Q_5+4Q_6+4Q_7+2Q_8+0Q_9+2Q_{10})(n\text{-}1) \\
Q_6(n) &= (0Q_1+4Q_2+2Q_3+2Q_4+2Q_5+4Q_6+2Q_7+2Q_8+4Q_9+1Q_{10})(n\text{-}1) \\
Q_7(n) &= (0Q_1+2Q_2+2Q_3+4Q_4+2Q_5+2Q_6+4Q_7+1Q_8+4Q_9+2Q_{10})(n\text{-}1) \\
Q_8(n) &= (2Q_1+0Q_2+4Q_3+2Q_4+2Q_5+4Q_6+2Q_7+3Q_8+4Q_9+2Q_{10})(n\text{-}1) \\
Q_9(n) &= (0Q_1+2Q_2+4Q_3+2Q_4+0Q_5+4Q_6+4Q_7+2Q_8+4Q_9+3Q_{10})(n\text{-}1) \\
Q_{10}(n) &= (2Q_1+2Q_2+0Q_3+4Q_4+2Q_5+2Q_6+4Q_7+2Q_8+6Q_9+3Q_{10})(n\text{-}1)
\end{aligned}$$

Representing them in matrix form:

$$\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \\ Q_9 \\ Q_{10} \end{pmatrix}(n) = \begin{pmatrix} 3&4&4&4&2&0&0&2&0&2 \\ 2&4&2&2&2&4&2&0&2&1 \\ 2&2&4&2&1&2&2&2&4&0 \\ 2&2&2&4&0&2&4&1&2&2 \\ 2&4&2&0&3&4&4&2&0&2 \\ 0&4&2&2&2&4&2&2&4&1 \\ 0&2&2&4&2&2&4&1&4&2 \\ 2&0&4&2&2&4&2&3&4&2 \\ 0&2&4&2&0&4&4&2&4&3 \\ 2&2&0&4&2&2&4&2&6&3 \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \\ Q_9 \\ Q_{10} \end{pmatrix}(n-1) = \begin{pmatrix} 3&4&4&4&2&0&0&2&0&2 \\ 2&4&2&2&2&4&2&0&2&1 \\ 2&2&4&2&1&2&2&2&4&0 \\ 2&2&2&4&0&2&4&1&2&2 \\ 2&4&2&0&3&4&4&2&0&2 \\ 0&4&2&2&2&4&2&2&4&1 \\ 0&2&2&4&2&2&4&1&4&2 \\ 2&0&4&2&2&4&2&3&4&2 \\ 0&2&4&2&0&4&4&2&4&3 \\ 2&2&0&4&2&2&4&2&6&3 \end{pmatrix}^n \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

We find the characteristic polynomial to be:
$|Q - xI| = (x+4)(x^9 - 40x^8 + 482x^7 - 2120x^6 - 79x^5 + 26128x^4 - 55636x^3 - 29184x^2 + 170880x - 112896)$, and the largest eigenvalue(spectral radius) to be[1]: $\rho(Q) \approx 22.936$,
Using matrix calculator[2] we find that:

$$Q(n) \approx (22.936)^n C_Q, \quad \text{where } C_Q^{\mathrm{T}} = [0.8728 \quad 0.8886 \quad 0.8994 \quad 0.9055 \quad 0.9937 \quad 1.0070 \quad 1.0130 \quad 1.1070 \quad 1.1150 \quad 1.2090]$$

Since the remaining eigen-values shrink to smaller values when its exponent **n** gets larger
Similar equations are obtained for King(K), Knight(Kn) and Bishop(B) respectively:

$$\begin{pmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \\ K_5 \\ K_6 \\ K_7 \\ K_8 \\ K_9 \\ K_{10} \end{pmatrix}(n) = \begin{pmatrix} 0&2&0&0&1&0&0&0&0&0 \\ 1&1&1&0&1&1&0&0&0&0 \\ 0&1&0&1&1&1&1&0&0&0 \\ 0&0&1&1&0&1&2&0&0&0 \\ 1&2&2&0&0&2&0&1&0&0 \\ 0&1&1&1&1&1&1&1&1&0 \\ 0&0&1&2&0&1&1&1&2&0 \\ 0&0&0&0&1&2&2&0&2&1 \\ 0&0&0&0&0&1&2&1&2&2 \\ 0&0&0&0&0&0&0&1&4&3 \end{pmatrix}^n \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$|K - \lambda I| = (\lambda - 3)\left(\lambda^9 - 6\lambda^8 - 27\lambda^7 + 78\lambda^6 + 342\lambda^5 + 216\lambda^4 - 252\lambda^3 - 243\lambda^2 + 27\right)$$

$$K(n) \approx (7.29086)^n C_K,$$

$$\text{where } C_K^{\mathrm{T}} = [0.18 \quad 0.35 \quad 0.47 \quad 0.53 \quad 0.66 \quad 0.88 \quad 1.00 \quad 1.19 \quad 1.35 \quad 1.54]$$

$$\begin{pmatrix} \mathrm{Kn}_1 \\ \mathrm{Kn}_2 \\ \mathrm{Kn}_3 \\ \mathrm{Kn}_4 \\ \mathrm{Kn}_5 \\ \mathrm{Kn}_6 \\ \mathrm{Kn}_7 \\ \mathrm{Kn}_8 \\ \mathrm{Kn}_9 \\ \mathrm{Kn}_{10} \end{pmatrix}(n) = \begin{pmatrix} 0&0&0&0&0&2&0&0&0&0 \\ 0&0&1&0&0&0&0&1&1&0 \\ 0&1&0&0&0&1&1&0&1&0 \\ 0&0&0&0&1&1&0&1&1&0 \\ 0&0&0&2&0&0&0&0&2&0 \\ 1&0&1&1&0&0&1&0&1&1 \\ 0&1&1&0&0&1&0&1&1&1 \\ 0&2&0&2&0&0&2&0&0&2 \\ 0&0&1&1&1&1&1&0&2&1 \\ 0&0&0&0&0&2&2&2&2&0 \end{pmatrix}^n \begin{pmatrix} 1\\1\\1\\1\\1\\1\\1\\1\\1\\1 \end{pmatrix}$$

$$|Kn - \lambda I| = (\lambda+2)\left(\lambda^9 - 4\lambda^8 - 23\lambda^7 + 53\lambda^6 + 104\lambda^5 - 181\lambda^4 - 142\lambda^3 + 192\lambda^2 + 52\lambda - 48\right)$$

$$\mathrm{Kn}(n) \approx (6.032847)^n C_{Kn},$$

$$\text{where } {C_{Kn}}^{\mathrm{T}} = [0.298 \quad 0.528 \quad 0.625 \quad 0.685 \quad 0.668 \quad 0.901 \quad 1.011 \quad 1.228 \quad 1.332 \quad 1.482]$$

$$\begin{pmatrix} \mathrm{B}_1 \\ \mathrm{B}_2 \\ \mathrm{B}_3 \\ \mathrm{B}_4 \\ \mathrm{B}_5 \\ \mathrm{B}_6 \\ \mathrm{B}_7 \\ \mathrm{B}_8 \\ \mathrm{B}_9 \\ \mathrm{B}_{10} \end{pmatrix}(n) = \begin{pmatrix} 1&0&0&0&2&0&0&2&0&2 \\ 0&2&0&0&0&2&0&0&2&1 \\ 0&0&2&0&1&0&2&0&2&0 \\ 0&0&0&2&0&2&2&1&0&0 \\ 2&0&2&0&1&0&0&2&0&2 \\ 0&2&0&2&0&2&0&0&2&1 \\ 0&0&2&2&0&0&2&1&2&0 \\ 2&0&0&2&2&0&2&1&0&2 \\ 0&2&2&0&0&2&2&0&2&1 \\ 2&2&0&0&2&2&0&2&2&1 \end{pmatrix}^n \begin{pmatrix} 1\\1\\1\\1\\1\\1\\1\\1\\1\\1 \end{pmatrix}$$

$$|B - \lambda I| = (\lambda+2)^3\left(\lambda^7 - 22\lambda^6 + 166\lambda^5 - 468\lambda^4 + 113\lambda^3 + 1066\lambda^2 - 184\lambda - 192\right)$$

$$\mathrm{B}(n) \approx (6.032847)^n C_B,$$

$$\text{where } {C_B}^{\mathrm{T}} = [0.943 \quad 0.795 \quad 0.719 \quad 0.687 \quad 1.085 \quad 0.945 \quad 0.888 \quad 1.253 \quad 1.146 \quad 1.511]$$

Finding no. of walks for chess pieces with fixed start-end points (note that the result below gives assumed average and further work and elaboration needs to be done to take the average result at face value):

Let $U = [4 \quad 8 \quad 8 \quad 8 \quad 4 \quad 8 \quad 8 \quad 4 \quad 8 \quad 4]$, be the matrix corresponding to no. of squares for each U_i where $i \in \{1,\cdots,10\}$,For any matrix $\mathrm{M} = \mathrm{Q,K,Kn}$ or B with fixed starting and ending points($M_{i,j}$ being we have: $\mathrm{M}_{i,j}(n) \approx C_{\mathrm{M}_i} C_{\mathrm{M}_j} (UC_{\mathrm{M}})^{-1} (22.936)^n$, where $i, j \in \{1,\cdots,10\}$ Where $C_{\mathrm{M}}(UC_{\mathrm{M}})^{-1}$ is the respective probabilities for fixed either starting or ending points for the matrix M(n) for large values of n.

Rook:

The problem will be split into two parts: 1) No. of ways a rook can reach H8 in n moves starting from A1 (8x8 board) , 2D walk of a rook is covered in detail in other papers[3] as well.

2) In a k dimensional board of the form: $L_1 \times L_2 \times \ldots \times L_k$, find the no. of ways a rook can traverse the board with a fixed starting and end points, use parameters for distinguishing the two,

Two ways to solve the first one: 1) Using Matrices and simultaneous recursive functions 2) Solve for a single dimension.

Assume $R_1(n)$ as no. of ways a rook can reach two fixed squares in 8 by 8 board that are not adjacent (sharing row or column), $R_2(n)$ as no. of ways a rook can reach two fixed squares in 8 by 8 board that share the same row or column, $R_3(n)$ as no. of ways a rook can reach its original square from where it started, we get equations:

$$R_1(n) = (14-2)R_1(n-1) + 2R_2(n-1) \qquad R_1(n) = 12R_1(n-1) + 2R_2(n-1) + 0R_3(n-1)$$

$$R_2(n) = 7R_1(n-1) + (7-1)R_2(n-1) + 1R_3(n-1) \qquad R_2(n) = 7R_1(n-1) + 6R_2(n-1) + 1R_3(n-1)$$

$$R_3(n) = 14R_2(n-1) \qquad R_3(n) = 0R_1(n-1) + 14R_2(n-1) + 0R_3(n-1)$$

The above equations can be written and solved in matrix format as:

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}(n) = \begin{bmatrix} 12 & 2 & 0 \\ 7 & 6 & 1 \\ 0 & 14 & 0 \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}(n-1)$$

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}(n) = \begin{bmatrix} 12 & 2 & 0 \\ 7 & 6 & 1 \\ 0 & 14 & 0 \end{bmatrix}^n \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}(0)$$

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}(n) = \begin{bmatrix} 1 & -1/7 & 1/49 \\ 1 & 3/7 & -1/7 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 14 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -2 \end{bmatrix}^n \frac{1}{64} \begin{bmatrix} 49 & 14 & 1 \\ -98 & 84 & 14 \\ 49 & -98 & 49 \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}(0)$$

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}(n) = \frac{1}{64} \begin{bmatrix} 49\times 14^n + 14\times 6^n + (-2)^n & 14\times 14^n - 12\times 6^n - 2\times(-2)^n & 14^n - 2\times 6^n + (-2)^n \\ 49\times 14^n - 42\times 6^n - 7\times(-2)^n & 14\times 14^n + 36\times 6^n + 14\times(-2)^n & 14^n + 6\times 6^n - 7\times(-2)^n \\ 49\times 14^n - 98\times 6^n + 49\times(-2)^n & 14\times 14^n + 84\times 6^n - 98\times(-2)^n & 14^n + 14\times 6^n + 49\times(-2)^n \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}(n) = 2^{n-6} \begin{bmatrix} 7^n - 2\times 3^n + (-1)^n \\ 7^n + 6\times 3^n - 7\times(-1)^n \\ 7^n + 14\times 3^n + 49\times(-1)^n \end{bmatrix}$$

2)Single dimension method:

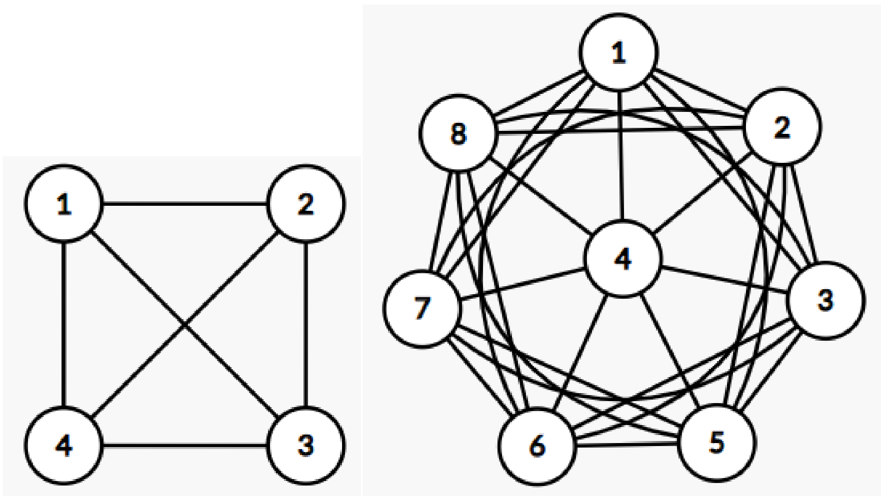
We solve for a k-dimensional board of the form: $L_1 \times L_2 \times ... \times L_k$, finding the no. of ways a rook can traverse the board with fixed starting and end points, we use parameters for distinguishing the two: $\theta_i = 1$ if the ith component of starting and ending point is same and $\theta_i = 0$ otherwise, for $i \in \{1, \cdots, k\}$. (For example, take starting point as p and ending point as q which can be from 1 to L_i if p=q then $\theta_i = 1$ otherwise its 0)

We now solve for a single dimension first:

If each square 1 to L represents a vertex of a Rook's graph (where two vertices are adjacent if a rook can move there in one move), then every vertex 1 to L is adjacent to every other vertex making it a complete graph (K_L)

Result 1:For a graph K_N the adjacency matrix is(where J is matrix of ones): $J - I$, walks of k-length will have the matrix as: $(J-I)^k = \left(\frac{(N-1)^k - (-1)^k}{N}\right) J + (-1)^k I$ for closed walk take i-ith entry of the matrix:

$\frac{1}{N}\left[(N-1)^k + (N-1)(-1)^k\right]$



(K$_4$ and K$_8$ graph for L=4 and 8 respectively)

Let θ represent the parameter whether the walk is closed or not, θ =0 for open walk and θ =1 for closed walk, for the matrix (J-I)^n if i=j then the i-jth entry will be: $\frac{1}{L}\left[(L-1)^n + (L-1)(-1)^n\right]$ (θ =1) for i≠j the entry will be: $\frac{1}{L}\left[(L-1)^n - (-1)^n\right]$ (θ =0), so for any walk with fixed end points it will be: $\frac{1}{L}\left[(L-1)^n + (\theta L - 1)(-1)^n\right]$

For example(L=4):

$$\begin{bmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{bmatrix}^n=\frac{1}{4}\begin{bmatrix}3^n+3(-1)^n&3^n-(-1)^n&3^n-(-1)^n&3^n-(-1)^n\\3^n-(-1)^n&3^n+3(-1)^n&3^n-(-1)^n&3^n-(-1)^n\\3^n-(-1)^n&3^n-(-1)^n&3^n+3(-1)^n&3^n-(-1)^n\\3^n-(-1)^n&3^n-(-1)^n&3^n-(-1)^n&3^n+3(-1)^n\end{bmatrix}$$

For a rectangular board ($L_1 \times L_2$), let the walk function(s) be: $F_i(n)=\frac{1}{L_i}\left[(L_i-1)^n+(\theta_i L_i-1)(-1)^n\right]$ for i=1,2 respectively, if the walk in horizontal is r-length and vertical is (n-r)-length, then the no. of such walks for the rook is: $\frac{n!}{n-r!r!}F_1(n-r)F_2(r)$ (since the rook can either move in the horizontal or vertical component at a time in any order)

So, the no. of such walks in a rectangle will be:

$$R_{2D}(n)=\sum_{r=0}^{n}{}^nC_rF_1(n-r)F_2(r)=\frac{1}{L_1L_2}\sum_{r=0}^{n}{}^nC_r\left[(L_1-1)^{n-r}+(\theta_1L_1-1)(-1)^{n-r}\right]\left[(L_2-1)^r+(\theta_2L_2-1)(-1)^r\right]$$

$$=\frac{1}{L_1L_2}\left[\left(L_1-1+L_2-1\right)^n+(\theta_2L_2-1)(L_1-2)^n+(\theta_1L_1-1)(L_2-2)^n+(-2)^n((\theta_1L_1-1)(\theta_2L_2-1))\right]$$

For $L_1 = L_2 = 8$ and $\theta_1 = \theta_2 = 0$ we have (no. of A1-H8 walks),

$$R_{2D}(n)=\frac{1}{64}\left[14^n-6^n-6^n+(-2)^n\right]=2^{n-6}\left[7^n-2\times 3^n+(-1)^n\right]$$

If we take $F_r(n)$ as a function for the permutations of the final position with parameters L_r and θ_r where r can be from 1 to k, and $R_{iD}(n)$ as a function of ways a rook can reach the final position in i-dimension space, then:

$$R_{2D}(n)=\sum_{r=0}^{n}{}^nC_rF_2(n-r)F_1(r)\quad R_{3D}(n)=\sum_{r_2=0}^{n}{}^nC_{r_2}F_3(n-r_2)\sum_{r_1=0}^{r_2}{}^{r_2}C_{r_1}F_2(r_2-r_1)F_1(r_1)\quad R_{kD}(n)=\sum_{r=0}^{n}{}^nC_rF_k(n-r)R_{(k-1)D}(r)$$

Through mathematical induction one can obtain (assume $\mathbb{N}_k=\{1,2,3,\ldots k\}$):

$$R_{kD}(n)=\sum_{S\subseteq\mathbb{N}_k}\left(\left(\prod_{r\in\mathbb{N}_k\setminus S}(\theta_rL_r-1)\right)\left(-k+\sum_{r\in S}L_r\right)^n\left(\prod_{r\in\mathbb{N}_k}L_r\right)^{-1}\right)\text{ and }R_{kD}(n)\approx\left(\sum_{r=1}^{k}L_r-k\right)^n\Big/\left(\prod_{r=1}^{k}L_r\right)\text{ as }n\to\infty$$

Let us introduce a concept of Combinatorial Binomial convolution, say:

$$f_i(n)=\sum_{\mu\in S_i}\left[\alpha_{i\mu}\left(\beta_{i\mu}\right)^n\right]\quad\forall i\in\mathbb{N}\text{ and }f_i*f_j(n)=\sum_{r=0}^{n}{}^nC_rf_i(n-r)f_j(r)=f_j*f_i(n)\quad\forall i,j\in\mathbb{N},$$

Then,

$$f_i*f_j(n)=\sum_{r=0}^{n}{}^nC_r\sum_{\mu\in S_i}\left[\alpha_{i\mu}\left(\beta_{i\mu}\right)^{n-r}\right]\sum_{\nu\in S_j}\left[\alpha_{j\nu}\left(\beta_{j\nu}\right)^r\right]=\sum_{\mu\in S_i,\nu\in S_j}\sum_{r=0}^{n}{}^nC_r\left[\alpha_{i\mu}\alpha_{j\nu}\left(\beta_{i\mu}\right)^{n-r}\left(\beta_{j\nu}\right)^r\right]=\sum_{\mu\in S_i,\nu\in S_j}\alpha_{i\mu}\alpha_{j\nu}\left(\beta_{i\mu}+\beta_{j\nu}\right)^n$$

and similarly, we get:

$$f_i*(f_j*f_k)(n)=\sum_{\mu_1\in S_i,\mu_2\in S_j,\mu_3\in S_k}\alpha_{i\mu_1}\alpha_{j\mu_2}\alpha_{k\mu_3}\left(\beta_{i\mu_1}+\beta_{j\mu_2}+\beta_{k\mu_3}\right)^n=(f_i*f_j)*f_k(n)=(f_i*f_j*f_k)(n)$$

Inductively we can get:

$$(f_1*f_2\cdots*f_k)(n)=\sum_{\mu_1\in S_i,\cdots,\mu_k\in S_k}\alpha_{1\mu_1}\cdots\alpha_{k\mu_k}\left(\beta_{1\mu_1}+\cdots+\beta_{k\mu_k}\right)^n$$

Now to get the above result simply substitute: $S_i=\{1,2\},\alpha_{i1}=\frac{1}{L_i},\alpha_{i2}=\frac{1}{L_i}(\theta_iL_i-1),\beta_{i1}=L_i-1,\beta_{i2}=-1$ to get:

$$F_i(n)=f_i(n)=\sum_{\mu\in S_i}\left[\alpha_{i\mu}\left(\beta_{i\mu}\right)^n\right]=\frac{1}{L_i}\left[(L_i-1)^n+(\theta_iL_i-1)(-1)^n\right]\text{ and}$$

$$(F_1*F_2\cdots*F_k)(n)=R_{kD}(n)=\sum_{S\subseteq\mathbb{N}_k}\left(\left(\prod_{r\in\mathbb{N}_k\setminus S}(\theta_rL_r-1)\right)\left(-k+\sum_{r\in S}L_r\right)^n\left(\prod_{r\in\mathbb{N}_k}L_r\right)^{-1}\right)$$

Assume it is true for k=t, we know that its true for k=1,

$$R_{1D}(n)=\sum_{S\subseteq\{1\}}\left(\left(\prod_{r\in\{1\}\setminus S}(\theta_r L_r-1)\right)\left(-1+\sum_{r\in S}L_r\right)^n\left(\prod_{r\in\{1\}}L_r\right)^{-1}\right)=\frac{1}{L_1}\left((L_1-1)^n+(\theta_1 L_1-1)(-1)^n\right)$$

$$R_{(t+1)D}(n)=\sum_{r=0}^{n}\left({}^nC_r\sum_{S\subseteq\mathbb{N}_t}\left(\left(\prod_{j\in\mathbb{N}_t\setminus S}\left(\theta_j L_j-1\right)\right)\left(-t+\sum_{j\in S}L_j\right)^r\left(\prod_{j\in\mathbb{N}_t}L_j\right)^{-1}\right)\frac{1}{L_{t+1}}\left(\left(L_{t+1}-1\right)^{n-r}+\left(\theta_{t+1}L_{t+1}-1\right)(-1)^{n-r}\right)\right)$$

$$=\left(\prod_{j\in\mathbb{N}_{t+1}}L_j\right)^{-1}\sum_{S\subseteq\mathbb{N}_t}\left(\left(\prod_{j\in\mathbb{N}_t\setminus S}\left(\theta_j L_j-1\right)\right)\sum_{r=0}^{n}\left({}^nC_r\left(-t+\sum_{j\in S}L_j\right)^r\left(\left(L_{t+1}-1\right)^{n-r}+\left(\theta_{t+1}L_{t+1}-1\right)(-1)^{n-r}\right)\right)\right)$$

$$=\left(\prod_{j\in\mathbb{N}_{t+1}}L_j\right)^{-1}\sum_{S\subseteq\mathbb{N}_t}\left(\left(\prod_{j\in\mathbb{N}_t\setminus S}\left(\theta_j L_j-1\right)\right)\left(\left(-t+\sum_{j\in S}L_j+L_{t+1}-1\right)^n+\left(\theta_{t+1}L_{t+1}-1\right)\left(-t-1+\sum_{j\in S}L_j\right)^n\right)\right)$$

$$=\left(\prod_{j\in\mathbb{N}_{t+1}}L_j\right)^{-1}\sum_{S\subseteq\mathbb{N}_t}\left(\left(\prod_{j\in\mathbb{N}_t\setminus S}\left(\theta_j L_j-1\right)\right)\left(-(t+1)+\sum_{j\in S\cup\{t+1\}}L_j\right)^n+\left(\prod_{j\in\mathbb{N}_{t+1}\setminus S}\left(\theta_j L_j-1\right)\right)\left(-(t+1)+\sum_{j\in S}L_j\right)^n\right)$$

$$=\sum_{S\subseteq\mathbb{N}_{t+1}}\left(\left(\prod_{j\in\mathbb{N}_{t+1}\setminus S}\left(\theta_j L_j-1\right)\right)\left(-t+\sum_{j\in S}L_j\right)^r\left(\prod_{j\in\mathbb{N}_{t+1}}L_j\right)^{-1}\right)$$

Thus, the result is proved by both induction and the convolution operation, the approximation result is obtained by simply ignoring terms that contain only smaller eigenvalues in magnitude as an exponent.

Additional Results and Scope for further problems

1)The solutions for number of Pawn-walks in 8x8 board is simply a combination of no. of walks for other four pieces: Rook, Knight, Queen and Bishop since a pawn must take 5-6 steps to covert to any of the four after which it simply copies other pieces, if P_1 represents pawns in A2,H2; P_2 in: B2,G2; P_3 in: C2,F2; P_4 in: D2,E2;

then we get: $$P_i(n)=\begin{cases}1 & n=0\\ 2 & 0<n\le 4\\ 5 & n=5\\ (Q_i+Kn_i+B_i+R)\circ(n-5)+(Q_i+Kn_i+B_i+R)\circ(n-6) & n\ge 6\end{cases}$$ for i=1,2,3 or 4

2)Solutions for no. of walks for remaining 4 pieces(King, Queen, Knight & Bishop) in other 2D square boards can be made by similarly finding matrix equations like it was done for 8x8, non-square boards and higher dimensional boards solutions are more complex since the pattern to obtain the matrices or walk recurrences are yet to be obtained, for instance a 16x16x16 and a 30x30 board will have at least 120 different functions with recursive relations, which require a matrix with at least 14,400 entries to solve!

3)Additional scope for walk-related problems can be made with different board configurations or structural variations, including various obstructions/other pieces/ tunnels/ conditional unit of k-dim board.
For k-dim board, introducing new piece variants (Assume board is hypercube with dim= L^k for convenience):
$\text{Board}=\{(x_1,\cdots,x_k):x_1,\cdots,x_k\in\mathbb{N}_L\}$
j-Rook: A piece that moves in exactly j components out of k, in 2D 1-rook is original rook and 2-rook is original bishop,
Say j=3, k=4 and $m\in\{1,2,3\}$) if $\vec{y}\in\text{Board}$, then valid moves are:

$(x_1, x_2, x_3, x_4) \to (x_1 \pm m, x_2 \pm m, x_3 \pm m, x_4) = \vec{y}$
$(x_1, x_2, x_3, x_4) \to (x_1, x_2 \pm m, x_3 \pm m, x_4 \pm m) = \vec{y}$
$(x_1, x_2, x_3, x_4) \to (x_1 \pm m, x_2, x_3 \pm m, x_4 \pm m) = \vec{y}$
$(x_1, x_2, x_3, x_4) \to (x_1, x_2 \pm m, x_3 \pm m, x_4 \pm m) = \vec{y}$

For k=2 and j=2 we simply have: $(x_1, x_2) \to (x_1 + m, x_2 + m), (x_1 + m, x_2 - m), (x_1 - m, x_2 + m), (x_1 - m, x_2 - m) = \vec{y}$

For k=2 and j=1 we have the rook: $(x_1, x_2) \to (x_1 \pm m, x_2), (x_1, x_2 \pm m) = \vec{y}$

j-Queen: A piece that can move at most j components out of k, in 2D 2-Queen is regular queen, the valid moves of j-Queen is simply a combination of all valid moves of j-Rooks, (j - 1)-Rooks, (j - 2)-Rooks up to 1-Rook.

No. of 1-moves from the corner for a j-Rook $= {}^kC_j(L-1)$

No. of 1-moves from the center for a j-Rook $= \begin{cases} {}^kC_j 2^j \dfrac{L-1}{2} & L \in \text{Even} \\ {}^kC_j 2^j \dfrac{L-2}{2} + {}^kC_j(1) & L \in \text{Odd} \end{cases} = {}^kC_j \left(2^j \left[\dfrac{L-1}{2} \right] + \dfrac{1+(-1)^L}{2} \right)$

No. of 1-moves from the corner for a j-Queen $= \sum_{r \in \mathbb{N}_j} {}^kC_r(L-1)$

No. of 1-moves from the center for a j-Queen $= \sum_{r \in \mathbb{N}_j} {}^kC_r \left(2^j \left[\dfrac{L-1}{2} \right] + \dfrac{1+(-1)^L}{2} \right)$

Extended knight pieces can be made but possibilities are limitless.

4)For non-walk related there is also placement related for eg: for placing N rooks in NxN board so that all the rooks are non-attacking we have N! ways, for placing N^2 rooks in NxNxN board we have $\prod_{r=1}^{N} r!$ ways, for queen and bishop the solution again gets complex, placement-related problems also depends on piece movement like walk-related problems.

Conclusion

In conclusion, this thesis delves into the realm of chess combinatorics, exploring the intricate patterns and possibilities inherent in the movements of various chess pieces on an 8x8 standard chessboard and beyond. Through rigorous calculations and application of combinatorial concepts such as the Binomial Theorem, Graph Theory, and Matrix Algebra, we have investigated the diverse ways in which pieces like the king, queen, knight, bishop, and rook can traverse the board. By representing these movements as graphs with vertices and edges, we have uncovered fascinating insights into the complexities of chess piece traversals, leveraging tools like adjacency matrices and recursive equations to derive solutions. Our exploration extends beyond the confines of the traditional 8x8 board, venturing into the realm of higher-dimensional boards and non-standard configurations.

Our findings contribute to both theoretical understanding and practical applications in chess and mathematics. We have highlighted the potential for further research, including investigating alternative board configurations, addressing placement-related challenges, and exploring more complex scenarios involving non-attacking arrangements of pieces. Overall, this thesis underscores the depth and relevance of chess combinatorics as a field of study.

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